Infinite randomness and quantum Griffiths effects in a classical system: The randomly layered Heisenberg magnet

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We investigate the phase transition in a three-dimensional classical Heisenberg magnet with planar defects, i.e., disorder perfectly correlated in two dimensions. By applying a strong-disorder renormalization group, we show that the critical point has exotic infinite-randomness character. It is accompanied by strong power-law Griffiths singularities. We compute various thermodynamic observables paying particular attention to finite-size effects relevant for an experimental verification of our theory. We also study the critical dynamics within a Langevin equation approach and find it extremely slow. At the critical point, the autocorrelation function decays only logarithmically with time while it follows a nonuniversal power law in the Griffiths phase.

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I. INTRODUCTION

The influence of impurities, defects, or other types of quenched disorder on the properties of phase transitions has aroused the interest of physicists for more than three decades (see Ref. 1 for an overview of some of the early work). Recently, this field has reattracted considerable attention as it has become clear that disorder effects are generically much stronger at zero-temperature quantum phase transitions than at classical thermal phase transitions. This leads to unconventional phenomena such as power-law quantum Griffiths singularities,[2–4] infinite-randomness critical points with exponential rather than power-law scaling,[5,6] or even smeared phase transitions.[7,8] A recent review of part of this physics can be found in Ref. 9.

The main reason for the enhanced disorder effects at quantum phase transitions is that the disorder is perfectly correlated in *imaginary time* direction. Because imaginary time acts as an extra dimension at a quantum phase transition (and becomes infinitely extended at zero temperature), one is effectively dealing with defects that are "infinitely large" in this extra dimension. Thus, they are much harder to average out than conventional finite-size defects.

This implies that similarly strong effects can be expected at a classical thermal phase transition if the disorder is perfectly correlated in one or more *space* dimensions. Indeed, it has been known for a long-time that the McCoy-Wu model, a classical two-dimensional Ising model with disorder perfectly correlated in one of the two dimensions, exhibits an unusual phase transition. In a series of papers,[10–13] McCoy and Wu developed a transfer-matrix approach to this model and showed that the specific heat is smooth across the ferromagnetic phase transition while the susceptibility is infinite over an entire temperature range. Fisher[5,6] later achieved an essentially complete understanding of this transition by means of a strong-disorder renormalization group (using the equivalence between the McCoy-Wu model and the one-dimensional random transverse-field Ising chain). He found that the critical point is of infinite-randomness type, and it is accompanied by strong power-law Griffiths singularities. Largely due to the fact that the McCoy-Wu model is difficult to realize in nature, these predictions have (to the best of our knowledge) not been experimentally verified yet.

In this paper, we present another classical system exhibiting an exotic infinite-randomness critical point, viz., a randomly layered three-dimensional (3D) Heisenberg magnet. This system is more easily realizable in experiment than the McCoy-Wu model as it can be produced by depositing random layers of two different ferromagnetic materials. Moreover, because of its three-dimensional character, it permits bulk thermodynamic measurements. We investigate the phase transition in this model by means of a strong-disorder renormalization group which allows us to determine the critical behavior exactly.

Our paper is organized as follows: in Sec. II, we introduce the randomly layered Heisenberg model and give heuristic arguments for the strong disorder effects. In Sec. III we explain our theoretical approach. Results on the thermodynamics are given in Sec. IV, while the experimentally important finite-size effects are discussed in Sec. V. Section VI is devoted to the dynamical behavior at the phase transition. We conclude in Sec. VII.

II. RANDOMLY LAYERED HEISENBERG MODEL

We consider a three-dimensional Heisenberg ferromagnet consisting of a random sequence of layers made of two different ferromagnetic materials, as sketched in Fig. 1.

This system can be modeled by a classical Heisenberg Hamiltonian on a cubic lattice given by

$$H = -\sum_{\mathbf{r}} J_z^{\parallel}(\mathbf{S}_{\mathbf{r}} \cdot \mathbf{S}_{\mathbf{r}+\hat{\mathbf{x}}} + \mathbf{S}_{\mathbf{r}} \cdot \mathbf{S}_{\mathbf{r}+\hat{\mathbf{y}}}) - \sum_{\mathbf{r}} J_z^{\perp} \mathbf{S}_{\mathbf{r}} \cdot \mathbf{S}_{\mathbf{r}+\hat{\mathbf{z}}}. \quad (1)$$

Here, $\mathbf{S}_{\mathbf{r}}$ is a three-component unit vector on lattice site $\mathbf{r}$ and $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are the unit vectors in the coordinate directions. The exchange interactions within the layers, $J_z^{\parallel}$, and between the layers, $J_z^{\perp}$, are both positive and independent random functions of the perpendicular coordinate z.

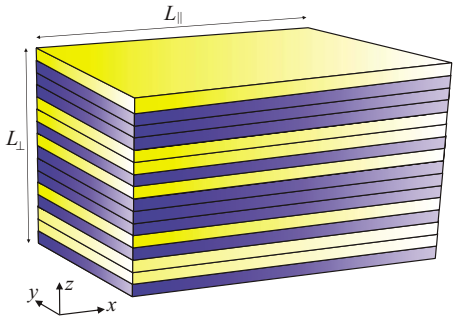

FIG. 1. (Color online) Schematic of the layered magnet: layers of two different ferromagnetic materials are arranged in a random sequence.

To develop a heuristic understanding of the randomly layered Heisenberg model, we first consider the case of all $J_z^\perp$ being identical, $J_z^\perp \equiv J^\perp$, while the $J_z^\parallel$ are drawn from a binary probability distribution

$$P(J^\parallel) = (1-p)\,\delta(J^\parallel - J_u) + p\,\delta(J^\parallel - J_l), \tag{2}$$

with $J_u > J_l$. Here, p is the concentration of the "weak" layers while $1-p$ is the concentration of the "strong" layers. More general distributions will be considered in the next section.

Let us now discuss the behavior of the model [Eq. (1)] qualitatively (see Fig. 2). At sufficiently high temperatures, the system will be in a conventional (strongly disordered) paramagnetic phase with a finite magnetic susceptibility which increases upon lowering the temperature. Below a temperature T_u (which is the transition temperature of a hypothetical system containing strong layers only, $J_z^\parallel \equiv J_u$), rare thick slabs of strong layers develop local order while the bulk system is still nonmagnetic. This is the weakly disordered Griffiths phase. The Griffiths phase continues below the actual critical temperature T_c down to a temperature T_l (which is the transition temperature of a hypothetical system containing weak layers only, $J_z^\parallel \equiv J_l$). In the weakly ordered Griffiths phase, bulk magnetism coexists with locally nonmagnetic slabs. Finally, below T_l, the system is in a conventional (strongly ordered) ferromagnetic phase.

To estimate the strength of the Griffiths singularities in this system, we need to compare the probability of finding a thick slab of strong layers with the contribution such a slab can make to thermodynamic quantities such as the susceptibility. Simple combinatorics gives the probability for finding a slab of L_{RR} consecutive strong layers to be

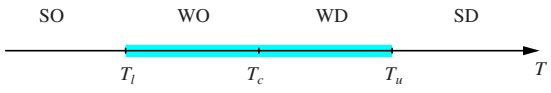

FIG. 2. (Color online) Schematic phase diagram of the randomly layered Heisenberg magnet [Eq. (1)]. SD and SO denote the conventional strongly disordered and strongly ordered phases, respectively. WD and WO are the weakly disordered and ordered Griffiths phases. T_c is the critical temperature while T_u and T_l mark the boundaries of the Griffiths phase.

$$w(L_{RR}) \sim (1-p)^{L_{RR}} = e^{-\tilde{p}L_{RR}}, \tag{3}$$

with $\tilde{p} = -\ln(1-p)$. Each such slab is equivalent to a two-dimensional Heisenberg model with an effective exchange interaction $L_{RR}J_u$. Because the two-dimensional Heisenberg model is exactly at its lower critical dimension, the susceptibility of the slab increases exponentially with the effective interaction,[9,14]

$$\chi(L_{RR}) \sim e^{bL_{RR}}, \tag{4}$$

where b increases with decreasing temperature. The same result also follows from a renormalization-group analysis of the corresponding nonlinear sigma model at its low-temperature fixed point[15] or from an explicit large-N calculation (as shown in the next section).

Thus, the exponential *decrease* in the rare region probability $w(L_{RR})$ with size L_{RR} is compensated by an exponential *increase* in the contribution it makes to the susceptibility. The total rare region susceptibility in the weakly disordered Griffiths phase is obtained by simply summing over the contributions of the individual rare regions. Up to pre-exponential factors, this yields

$$\chi_{RR} = \int dL_{RR} e^{(b-\tilde{p})L_{RR}}. \tag{5}$$

The total rare region susceptibility thus diverges once b becomes larger than $\tilde{p}$. Other observables can be discussed along the same lines. Equations (3)–(5) are analogous to the corresponding relations for the McCoy-Wu model[10–13] or (after quantum-to-classical mapping) to those of the random transverse-field Ising model.[2–4] This suggests that the phase transition in our model displays unconventional behavior. In the next section we investigate this question in detail by means of a renormalization-group method.

III. STRONG-DISORDER RENORMALIZATION GROUP

In this section we study the ferromagnetic phase transition of the randomly layered Heisenberg model by means of a strong-disorder renormalization group.[16,17] Our implementation of this method follows a recent study of dissipative quantum phase transitions.[18,19] We therefore only outline the major steps of the calculation, details can be found in Ref. 19.

A. Order-parameter field theory

Our starting point is a Landau-Ginzburg-Wilson (LGW) order-parameter field theory for an N-component order parameter $\phi(\mathbf{r})$. In the absence of disorder, the free-energy functional reads

$$S = \int d^3r [\,\delta_0 \phi^2(\mathbf{r}) + \gamma_0^2 (\partial_\mathbf{r}\phi(\mathbf{r}))^2 + u\phi^4(\mathbf{r})]. \tag{6}$$

Here, δ_0 is the bare distance from criticality, γ_0 is the bare length scale, and u is the ϕ^4 coefficient. In the presence of our layered disorder, δ_0, γ_0, and u become random functions of the z coordinate (the coordinate perpendicular to the lay-

ers) and the derivative term will generally be anisotropic. In order to apply the real-space-based strong-disorder renormalization group, we discretize the continuum LGW theory [Eq. (6)] in the z direction but not in the xy plane. For simplicity, we first consider the large-N limit of our LGW theory which allows us to perform all of the following calculations explicitly. We will later show that the resulting critical point is the same for all $N>2$ including the physically relevant Heisenberg case $N=3$. The discrete large-N order parameter field theory reads as

$$S = \sum_{z,\mathbf{q}} (\delta_z + \lambda_z + \gamma_z^2 \mathbf{q}^2) |\phi_z(\mathbf{q})|^2 - \sum_{z,\mathbf{q}} J_z^\perp \phi_z(\mathbf{q}) \phi_{z+1}(-\mathbf{q}), \tag{7}$$

where $\mathbf{q}$ is a two-component vector describing the xy momentum. The Lagrange multipliers λ_z enforce the large-N constraints $\langle (\phi_z^{(k)})^2 \rangle = 1$ for the kth order-parameter component in layer z; they have to be determined self-consistently. The renormalized local distance from criticality in layer z is given by $\epsilon_z = \delta_z + \lambda_z$. In the disordered phase, *all* $\epsilon_z > 0$. For the case of a single layer, the LGW theory [Eq. (7)] can be solved immediately, giving

$$\epsilon_z = \gamma_z^2 \Lambda^2 e^{-4\pi\gamma_z^2/a^2}, \tag{8}$$

with Λ being a momentum cutoff and a as the lattice constant.

B. Recursion relations

The basic idea of the strong-disorder renormalization group is to successively integrate out local high-energy degrees of freedom. In the LGW theory [Eq. (7)], the competing local couplings are the local distances from criticality ϵ_z and the interactions $J_z^\perp$. In the bare theory, they are independent random variables with distributions $R_0(\epsilon)$ and $P_0(J^\perp)$, respectively. The method relies on these distributions being broad and becomes exact in the limit of infinitely broad distributions. We will verify this condition *a posteriori*.

In each renormalization-group step, we choose the largest local coupling $\Omega = \max\{\epsilon_z, J_z^\perp\}$. If it is a distance from criticality, say ϵ_2, the unperturbed part of the free energy is $S_0 = \Sigma_{\mathbf{q}} (\epsilon_2 + \gamma_2^2 \mathbf{q}^2) |\phi_2(\mathbf{q})|^2$. The coupling of ϕ_2 to the neighboring layers, $S_1 = -\Sigma_{\mathbf{q}} [J_1^\perp \phi_1(\mathbf{q}) \phi_2(-\mathbf{q}) + J_2^\perp \phi_2(\mathbf{q}) \phi_3(-\mathbf{q})]$, is treated perturbatively. Keeping only the leading long-wavelength terms that arise in second order of the cumulant expansion, we obtain renormalized interactions $\tilde{S} = -\Sigma_{\mathbf{q}} \tilde{J}_1^\perp \phi_1(\mathbf{q}) \phi_3(-\mathbf{q})$ with

$$\tilde{J}_1^\perp = \frac{J_1^\perp J_2^\perp}{\epsilon_2}. \tag{9}$$

At the end of the renormalization-group step, ϕ_2 is dropped from the action.

If the largest local energy is an interaction, say $J_2^\perp$, we solve the two-layer problem $S_0 = \Sigma_{\mathbf{q}} \Sigma_{z=2,3} (\epsilon_z + \gamma_z^2 \mathbf{q}^2) |\phi_z(\mathbf{q})|^2 - \Sigma_{\mathbf{q}} J_2^\perp \phi_2(\mathbf{q}) \phi_3(-\mathbf{q})$ exactly while treating the interactions with the neighboring layers as perturbations. For $J_2^\perp \gg \epsilon_2, \epsilon_3$, the two fields ϕ_2 and ϕ_3 are essentially parallel; thus they can be replaced by a single field $\tilde{\phi}_2$ with an effective renormalized free-energy functional $\tilde{S} = \Sigma_{\mathbf{q}} (\tilde{\epsilon}_2 + \tilde{\gamma}_2^2 \mathbf{q}^2) |\tilde{\phi}_2(\mathbf{q})|^2$.

After a straightforward but somewhat lengthy calculation,[19] the effective distance from criticality of the combined layer comes out to be

$$\tilde{\epsilon}_2 = 2 \frac{\epsilon_2 \epsilon_3}{J_2^\perp}, \tag{10}$$

while the length scale parameter renormalizes as $\tilde{\gamma}_2^2 = \gamma_2^2 + \gamma_3^2$. The renormalized field represents a layer with effective moment per site

$$\tilde{\mu}_2 = \mu_2 + \mu_3. \tag{11}$$

The interactions of the combined layer with the neighboring layers are not renormalized. The net result of the renormalization-group step is the elimination of one layer and the reduction in the energy scale Ω.

The structure of the renormalization-group recursion relations (9)–(11) is identical to those of the one-dimensional random transverse-field Ising model[5,6] as well as the dissipative quantum rotor model.[18,19] Consequently (and somewhat surprisingly), the thermal phase transition in our randomly layered classical three-dimensional Heisenberg model belongs to the same universality class as the quantum phase transitions in the one-dimensional random transverse-field Ising model and the dissipative quantum rotor chain.

At first glance, this result seems to suggest that crucial system characteristics such as order-parameter symmetry and dimensionality are rendered unimportant by the strong-disorder renormalization group. However, the physics turns out to be more subtle. The fact that our randomly layered Heisenberg model and the random transverse-field Ising chain are in the same universality class is due to a nontrivial interplay between the order parameter symmetry and the defect dimensionality. We will discuss this point in more detail in Sec. VII in the context of a general classification of phase transitions in the presence of disorder.

C. Fixed points

The renormalization-group step outlined in the last subsection does not change the lattice topology because we remove a full layer in each step. Moreover, the surviving ϵ and $J^\perp$ remain statistically independent. The theory can therefore be formulated in terms of individual probability distributions $P(J^\perp)$ and $R(\epsilon)$. Fisher derived flow equations for these distributions and solved them analytically.[5,6] They have three kinds of nontrivial fixed points representing the weakly ordered and disordered Griffiths phases as well as the critical point in between. At the critical fixed point, the relative width of the distributions $P(J^\perp)$ and $R(\epsilon)$ diverges, justifying the method and giving the critical point its name, *infinite-randomness* critical point.

The critical behavior is characterized by three exponents, $\nu = 2$, $\psi = 1/2$, and $\phi = (1+\sqrt{5})/2$. The exponent ν controls how the perpendicular correlation length $\xi_\perp$ diverges as the critical point is approached

$$\xi_\perp \sim |\delta|^{-\nu}. \tag{12}$$

$\xi_\perp$ characterizes the spatial correlations perpendicular to the layers (in z direction). δ is the fully renormalized distance from the critical point; it is given by $\delta \sim [\ln(\epsilon) - \ln(J^\perp)]_0$ in terms of the bare variables ($[\cdot]_0$ denotes the average over the bare disorder distributions.)

The exponent ψ (which is sometimes called the tunneling exponent because of its meaning in the quantum problem of Refs. 5 and 6) relates the perpendicular correlation length $\xi_\perp$ and the correlation length $\xi_\parallel$ within the layers. The scaling is highly anisotropic,

$$\ln(\xi_\parallel/a) \sim \xi_\perp^\psi. \tag{13}$$

ψ also controls the density n_Ω of layers surviving at energy scale Ω in the renormalization procedure. The scaling form of this variable is given by[19]

$$n_\Omega(\delta) = [\ln(\Omega_I/\Omega)]^{-1/\psi} X_n[\delta^{\nu\psi} \ln(\Omega_I/\Omega)], \tag{14}$$

where Ω_I is a constant of the order of the initial (bare) value of Ω. The scaling function behaves as $X_n(0)=$const and $X_n(y\to\infty) \sim y^{1/\psi} e^{-cy}$, where c is a constant. As a result, the layer density decreases as $n_\Omega \sim [\ln(\Omega_I/\Omega)]^{-1/\psi}$ at criticality while it behaves as $n_\Omega \sim \delta^\nu \Omega^{1/z}$ in the disordered Griffiths phase ($\delta>0$). The nonuniversal exponent z varies as $z \sim \delta^{-\nu\psi}$ in the Griffiths phase.

The exponent ϕ determines how the typical moment μ_Ω per site of a surviving layer depends on the energy scale Ω. The scaling form of μ_Ω reads as

$$\mu_\Omega(\delta) = [\ln(\Omega_I/\Omega)]^\phi X_\mu[\delta^{\nu\psi} \ln(\Omega_I/\Omega)]. \tag{15}$$

The scaling function behaves as $X_\mu(0)=$const and $X_\mu(y\to\infty) \sim y^{1-\phi}$. Thus, at criticality the typical moment increases as $\mu_\Omega \sim [\ln(\Omega_I/\Omega)]^\phi$ while it behaves as $\mu_\Omega \sim \delta^{\nu\psi(1-\phi)} \ln(\Omega_I/\Omega)$ in the disordered Griffiths phase.

IV. THERMODYNAMICS

The overall strategy[6] for computing the behavior of thermodynamic observables consists in running the strong-disorder renormalization group from the initial energy scale Ω_I down to the energy scale set by an external perturbation such as a magnetic field. The high-energy degrees of freedom eliminated in this way do not make significant contributions to the long-wavelength physics. The surviving layers are very weakly coupled and can be treated as independent. In this section we show that the resulting thermodynamic behavior of our system is similar to that at an infinite-randomness quantum critical point. However, there are significant differences due to the fact that we are dealing with a thermal (classical) phase transition.

A. Single-layer results

We start by considering a single layer with effective moment μ per site in an external magnetic field (i.e., this layer is the result of combining μ original layers during the renormalization group). The free-energy functional is given by

$$S_l = \sum_{\mathbf{q}} (\delta + \lambda + \gamma^2 \mathbf{q}^2) |\phi(\mathbf{q})|^2 - \mu \sum_{\mathbf{q}} h(\mathbf{q}) \phi(-\mathbf{q}), \tag{16}$$

where $h(\mathbf{q})$ is the Fourier transform of the external field at wave vector $\mathbf{q}$. This theory is Gaussian, thus the partition function and free energy are easily calculated. For a uniform magnetic field h, the free energy reads as $F_l(h) = \Sigma_{\mathbf{q}} \ln(\epsilon + \gamma^2 \mathbf{q}^2) - L_\parallel^2 \mu^2 h^2 / 4\epsilon$. Here $\epsilon = \delta + \lambda$ as before, and $L_\parallel$ is the linear size of the layer. The value of the Lagrange multiplier λ follows from the large-N constraint

$$\langle \phi^2 \rangle = \frac{1}{L_\parallel^2} \frac{\partial F_l}{\partial \epsilon} = \frac{1}{L_\parallel^2} \sum_{\mathbf{q}} \frac{1}{\epsilon + \gamma^2 \mathbf{q}^2} + \frac{\mu^2 h^2}{4\epsilon^2} = 1. \tag{17}$$

For small fields, $\mu h \ll \epsilon(h=0)$, the first term in the sum dominates, yielding $\epsilon(h) = \epsilon(0) + O(h^2)$ with $\epsilon(0)$ given by Eq. (8). In the opposite limit, $\mu h \gg \epsilon(h=0)$, the second term dominates, resulting in $\epsilon(h) = \mu h/2$.

The magnetization of the single layer is easily computed by taking the appropriate derivative of the free energy

$$m_l = -(1/L_\parallel^2)(\partial F_l / \partial h)_\epsilon = \mu^2 h / 2\epsilon, \tag{18}$$

and the zero-field uniform susceptibility is given by

$$\chi_l = \mu^2 / 2\epsilon(0). \tag{19}$$

Other observables can be computed in an analogous fashion. For instance, the local susceptibility $\chi_{l,loc}$ takes the same form as Eq. (19) with μ^2 replaced by μ.

B. Critical point and weakly disordered Griffiths phase

We now combine the single-layer observables with the strong-disorder renormalization-group results for the density [Eq. (14)] and the moment [Eq. (15)] of the surviving layers. In the present subsection we focus on the critical point and the disordered Griffiths phase while the ordered Griffiths phase will be addressed in Sec. IV C.

The total magnetization in a magnetic field h can be obtained by running the renormalization group to the energy scale $\Omega_h = \mu_{\Omega_h} h$. All the surviving layers have $\epsilon \ll \mu h$ and are thus fully polarized. The total magnetization per site thus reads as

$$\begin{aligned} m(\delta, h) &= n_{\Omega_h}(\delta) \mu_{\Omega_h}(\delta) \\ &= [\ln(\Omega_I/\Omega_h)]^{\phi - 1/\psi} \Theta_m[\delta^{\nu\psi} \ln(\Omega_I/\Omega_h)]. \end{aligned} \tag{20}$$

The scaling function is given by $\Theta_m(y) = X_n(y) X_\mu(y)$. Now, using the fact that $\Omega_h = \mu_{\Omega_h} h$, we find $m \sim [\ln(\Omega_I/h)]^{\phi - 1/\psi}$ (with double-logarithmic corrections) at criticality, $\delta = 0$. This implies that the critical isotherm exponent $\bar{\delta}$ (commonly defined via $m \sim h^{1/\bar{\delta}}$) is formally infinite. In the Griffiths phase, $\delta > 0$, we obtain $m \sim h^{1/z} \delta^{\nu + \nu\psi(1-\phi)(1+1/z)} [\ln(\Omega_I/h)]^{1+1/z}$. As long as $z \sim \delta^{-\nu\psi}$ is larger than one (i.e., sufficiently close to the critical point) this contribution dominates the regular linear-response term. We thus find a nonuniversal power-law singularity in a finite temperature interval around the critical point.

In the zero-field limit, the uniform susceptibility $\chi = \partial m / \partial h \sim h^{1/z - 1}$ consequently diverges not just at the criti-

cal point but for all $z>1$, again in an entire temperature range around the critical point. This result can also be obtained by summing Eq. (19) over all layers using the spectral density $\rho(\epsilon)=dn_\Omega/d\Omega|_{\Omega=\epsilon}$. In the Griffiths phase this gives the following rare region contribution to the susceptibility

$$\chi(h\to 0)\sim\begin{cases}\infty & (z>1)\\ \dfrac{z}{1-z}\Omega_I^{1/z-1} & (z<1).\end{cases} \tag{21}$$

The specific heat can be obtained by summing the single-layer free energy F_l over the spectral density $\rho(\epsilon)$ and taking the appropriate derivatives with respect to the reduced temperature. As in the McCoy-Wu model, the resulting specific heat is smooth across the transition.

C. Weakly ordered Griffiths phase

While the order parameter symmetry does not play a significant role on the disordered side of the critical point where all conventional (non rare-region) excitations are gapped, it becomes important on the ordered side of the transition where gapless excitations exist even in the absence of our rare region physics. This leads to some minor differences between our results and those of the McCoy-Wu model.

To determine the spontaneous magnetization on the ordered side of the transition, we follow the strong-disorder renormalization-group flow from Ω_I toward $\Omega=0$. For small but nonzero $|\delta|$, i.e., close to the critical point, the flow initially follows the critical trajectory until the renormalization-group length scale reaches the correlation length $\xi_\perp\sim|\delta|^{-\nu}$. This occurs at an energy Ω_ξ given by $\ln(\Omega_I/\Omega_\xi)\sim\xi_\perp^\psi\sim|\delta|^{-\nu\psi}$. Beyond this scale, the system is essentially ordered, and almost no layers will be removed under further action of the renormalization group.

We can therefore find the spontaneous magnetization by counting how many of the original layers survive at length scale $\xi_\perp$. This leads to

$$m\sim n_{\Omega_\xi}\mu_{\Omega_\xi}\sim[\ln(\Omega_I/\Omega_\xi)]^{\phi-1/\psi}\sim|\delta|^{\nu(1-\phi\psi)}. \tag{22}$$

The order-parameter critical exponent thus takes the value $\beta=\nu(1-\phi\psi)$. In a small magnetic field h, the magnetization picks up a nonanalytic correction which can be computed following the methods of Ref. 6. We find

$$m(h)-m(0)\sim h^{1/(1+z)}, \tag{23}$$

implying that the (longitudinal) susceptibility $\chi=\partial m/\partial h\sim h^{-z/(z+1)}$ diverges in the zero-field limit everywhere in the weakly ordered Griffiths phase. (The transverse susceptibility is infinite everywhere in the ordered phase simply because of the continuous order-parameter symmetry.)

Another important property of the ordered phase of a continuous symmetry magnet is the spin-wave stiffness which can be defined via the change in the free energy with a twist in the boundary conditions. In our system, we must distinguish the parallel spin-wave stiffness from the perpendicular one. To find the parallel spin-wave stiffness $\rho_s^\parallel$, we apply boundary conditions at $x=0$ and $x=L_\parallel$ such that the spins at the two ends are at a relative angle Θ. In the limit of small Θ and large $L_\parallel$, the free-energy density f depends on Θ as

$$f(\Theta)-f(0)=\frac{1}{2}\rho_s^\parallel\left(\frac{\Theta}{L_\parallel}\right)^2, \tag{24}$$

which defines $\rho_s^\parallel$.

In our system, the free-energy cost due to the twist is simply the sum over all layers participating in the long-range order. Each layer has the same twisted boundary conditions and the perpendicular bonds (which are not twisted) do not contribute. The bare stiffness of a single layer is given by γ^2. Because γ^2 is additive under the strong-disorder renormalization group, $\rho_s^\parallel$ behaves like the layer moment per site, $\rho_s^\parallel\sim\mu$. The calculation of the total parallel spin-wave stiffness thus proceeds analogously to the total magnetization yielding

$$\rho_s^\parallel\sim\gamma_0^2|\delta|^\beta=\gamma_0^2|\delta|^{\nu(1-\phi\psi)}. \tag{25}$$

If a global twist is applied perpendicular to the layers, i.e., between the bottom ($z=0$) and top ($z=L_\perp$) of the stack, the local twist Θ_z between layers z and $z+1$ will vary from layer to layer according to the local $J_z^\perp$. The total free-energy cost can be written as

$$f(\Theta)-f(0)\sim\frac{1}{2L_\perp}\sum_z\rho_z\Theta_z^2, \tag{26}$$

with $\rho_z\sim J_z^\perp$. Minimizing $f(\Theta)-f(0)$ under the constraint $\Sigma_z\Theta_z=\Theta$ gives $\Theta_z\sim 1/\rho_z$ and

$$f(\Theta)-f(0)\sim\left(L_\perp\sum_z\rho_z^{-1}\right)^{-1}. \tag{27}$$

To obtain an upper bound for $f(\Theta)-f(0)$, we estimate $\Sigma_z\rho_z^{-1}$ by its largest contribution, $\rho_{\min}^{-1}\sim(J_{\min}^\perp)^{-1}$. In the weakly ordered Griffiths phase, the fixed-point distribution of $J^\perp$ is gapless,[6] and $J_{\min}^\perp$ vanishes as $L_\perp^{-z}$ in the thermodynamic limit $L_\perp\to\infty$. We conclude $f(\Theta)-f(0)\sim L_\perp^{-1-z}$, implying that the global perpendicular stiffness vanishes, $\rho_s^\perp=0$ (for $z>1$). The weakly ordered Griffiths phase is thus very peculiar because the system displays long-range ferromagnetic order but it has no (perpendicular) spin-wave stiffness.

V. FINITE-SIZE EFFECTS

The results in Sec. IV were for an infinite system (in the thermodynamic limit). Here, we briefly discuss the effects of a finite system size in either parallel or perpendicular direction.

We start with a finite in-plane (parallel) size $L_\parallel$. It plays the same role as a finite temperature in the quantum phase transitions in Refs. 5, 6, 18, and 19 where the inverse temperature is the system size in imaginary time direction. Solving the large-N constraint for a single layer of linear size $L_\parallel$ gives $\epsilon(L_\parallel)=\epsilon(\infty)+O(1/L_\parallel^2)$ for $\epsilon(\infty)\gg 1/L_\parallel^2$. Here, $\epsilon(\infty)$ is the thermodynamic limit result given in Eq. (8). In the opposite limit, $\epsilon(\infty)\ll 1/L_\parallel^2$, we obtain $\epsilon(L_\parallel)=1/L_\parallel^2$. Thus, a finite $L_\parallel$ cuts off the low-ϵ tail in the spectral density $\rho(\epsilon)$.

As an example of the resulting finite-size effects in thermodynamic quantities we now discuss the dependence of the susceptibility on $L_\parallel$. Within the strong-disorder renormaliza-

tion group, it can be found by running the renormalization group to the scale $\Omega_L = 1/L_\parallel^2$. Beyond that scale, ϵ is not renormalized further down. All surviving layers now have $\epsilon \gg J^\perp$ and can thus be treated as independent. Using Eq. (19), the uniform susceptibility of a system of size $L_\parallel$ is consequently given as the sum over all layers surviving at scale Ω_L,

$$\chi(\delta, L_\parallel) = n_{\Omega_L}(\delta)\mu^2_{\Omega_L}(\delta)/2\Omega_L. \tag{28}$$

At criticality, $\delta=0$, this leads to $\chi \sim L_\parallel^2[\ln(L_\parallel/a)]^{2\phi-1/\psi}$. We emphasize that χ is the susceptibility *per volume*, so $L_\parallel^2$ is not simply a geometric factor but indicates the divergence of the susceptibility in the thermodynamic limit. In the weakly disordered Griffiths phase, the same calculation gives (up to logarithmic corrections) a nonuniversal power-law dependence, $\chi \sim \delta^{\nu+2\nu\psi(1-\phi)} L_\parallel^{2-2/z}$. In the weakly ordered Griffiths phase, we need to take into account that long-range order is not possible for any finite $L_\parallel$. Thus all layers surviving at scale Ω_L will again contribute to the susceptibility. In contrast to the weakly disordered Griffiths phase, the typical moment of a layer is proportional to its thickness $\mu_{\Omega_L} \sim m_0 n_{\Omega_L}^{-1}$, where m_0 is the bulk magnetization. In the weakly ordered Griffiths phase, we thus obtain $\chi \sim \delta^{\nu-\nu\phi\psi} L_\parallel^{2+2/z}$. All of our results for the $L_\parallel$ dependence of the uniform susceptibility are completely analogous to the corresponding temperature dependencies at the quantum phase transition in the random transverse-field Ising chain in Ref. 6. They are also compatible with finite-size scaling using that $1/L_\parallel^2$ scales like ϵ (or, equivalently, like a magnetic field H). Other observables can be worked out in a similar fashion.

We now turn to the effects of a finite size $L_\perp$ in perpendicular direction, i.e., the effects of a finite number of layers in our stack. We expect these effects to be particularly important experimentally because growing samples containing a macroscopic number of layers will often be difficult. The origin of finite-size effects in $L_\perp$ is that finite-size samples do not contain rare regions (strongly coupled layers) beyond a certain thickness or, equivalently, they do not contain rare regions with $\epsilon < \Omega_{\min}(L_\perp)$.

Within the strong-disorder renormalization group, the relation between system size and the cut-off energy scale $\Omega_{\min}(L_\perp)$ can be worked out using the density of surviving layers n_Ω. In a typical sample of size $L_\perp$, the *number* of layers surviving at renormalization-group scale Ω is given by $L_\perp n_\Omega$. The cut-off scale is thus defined by $L_\perp n_{\Omega_{\min}} = 1$. At criticality, this implies

$$\ln(\Omega_I/\Omega_{\min}) \sim L_\perp^\psi \tag{29}$$

reflecting the activated character of finite-size scaling in perpendicular direction. In the two Griffiths phases, we obtain

$$\Omega_{\min} \sim |\delta|^{-\nu z} L_\perp^{-z}. \tag{30}$$

As the first example of the resulting finite-size effects in thermodynamic quantities, we consider the magnetization-field curve $m(h)$. To do so, we compare the field-induced renormalization-group cutoff $\Omega_h = \mu_{\Omega_h} h$ and the finite-size cutoff $\Omega_{\min}$. As long as $\Omega_h > \Omega_{\min}$, the finite system size $L_\perp$ has only a negligible effect on the magnetization. However, $L_\perp$, cuts off the nonlinear low-field tail of $m(h)$ once $\Omega_h < \Omega_{\min}$. At criticality, this happens for fields below $h_{\min}$ given by $\ln(\Omega_I/h_{\min}) \sim L_\perp^\psi$. In the weakly disordered Griffiths phase, the nonlinear $m(h)$ curve is cut off below $h_{\min} \sim \delta^{-\nu z-\nu\psi(1-\phi)} L_\perp^{-z}$. In the weakly ordered Griffiths phase, the calculation is slightly more involved because we first need to resolve the relation between $\Omega_h = \mu_{\Omega_h} h$ using $\mu_{\Omega_h} \sim m_0 n_{\Omega_h}^{-1}$, where m_0 is the bulk magnetization. We finally obtain $h_{\min} \sim |\delta|^{\nu\phi\psi-\nu(1+z)} L_\perp^{-(1+z)}$.

The zero-field susceptibility of a typical sample of perpendicular size $L_\perp$ can be calculated by summing Eq. (19) over all layers with the spectral density $\rho(\epsilon)$ cut off at $\epsilon = \Omega_{\min}$. Alternatively, it can be estimated by $(\partial m/\partial h)_{h_{\min}}$. At criticality, the susceptibility diverges exponentially with system size, $\chi \sim \exp(AL_\perp^\psi)$ with A as a constant. In the weakly disordered Griffiths phase, we find a nonuniversal power law, $\chi \sim \delta^{1+\nu z+2\nu\psi(1-\phi)} L_\perp^{z-1}$.

Finally, we discuss the finite-size behavior of the perpendicular spin-wave stiffness $\rho_s^\perp$ in the weakly ordered Griffiths phase. In Sec. IV C, we showed that $\rho_s^\perp$ vanishes in the thermodynamic limit because the fixed-point distribution of $J^\perp$ is gapless. A finite perpendicular size $L_\perp$ establishes a lower bound for $J^\perp$ in a typical sample. From Eq. (30) we obtain $J^\perp_{\min} \sim L_\perp^{-z}$. Thus, the perpendicular spin-wave stiffness of a typical sample vanishes as $\rho_s^\perp \sim L_\perp^{1-z}$ with increasing system size (for $z>1$).

VI. CRITICAL DYNAMICS

It is well known that dynamic critical phenomena show stronger rare region effects and Griffiths singularities than the corresponding thermodynamic critical phenomena at classical phase transitions. In particular, rare regions dominate the long-time dynamics in a conventional classical Griffiths phase[20–23] even though they provide only small corrections to the thermodynamics. In this section, we therefore study the critical dynamics in our randomly layered Heisenberg magnet.

The classical Heisenberg model does not have any internal dynamics, we therefore add a phenomenological dynamics to our system. Here, we focus on the simplest case, a purely relaxational dynamics corresponding to model A in the classification of Hohenberg and Halperin.[24] Microscopically, this type of dynamics can be realized, e.g., via the Glauber[25] or Metropolis[26] algorithms. Other dynamical algorithms can be studied using similar methods (including model J which describes the dynamics of real Heisenberg spins). This remains a task for the future.

To characterize the dynamic critical behavior, we calculate the average autocorrelation function

$$C(t) = \frac{1}{L_\perp L_\parallel^2} \int d^3r \langle \phi(\mathbf{r},t)\phi(\mathbf{r},0)\rangle, \tag{31}$$

where $\phi(\mathbf{r},t)$ is the order parameter at position $\mathbf{r}$ and time t. In addition, we also determine the dynamic susceptibility $\chi(\omega)$.

Let us begin by considering the dynamics of a single layer with moment μ per site and a renormalized local distance ϵ

from criticality. Because a single layer cannot display long-range order, the correlations decay exponentially in time. The dependence of the correlation (relaxation) time ξ_t on ϵ can be found following the heuristic arguments of Bray.[22] He considered a correlation volume $\xi_\parallel^2 \sim 1/\epsilon$ which he assumed to be in the magnetic state with total magnetization $M_0 \sim \mu \xi_\parallel^2 \sim \mu/\epsilon$. The relaxation of the magnetization occurs mainly via diffusion of the order-parameter vector on a sphere of radius $|M_0|$ due to thermal noise (because there are no energy barriers in the case of Heisenberg symmetry). The noise at different points in space and time adds incoherently. Thus, according to the central limit theorem, the change in magnetization after time t is $\Delta M(t) \sim t^{1/2}(\mu/\epsilon)^{1/2}$. Defining ξ_t as the time when $\delta M(\xi_t) \sim M_0$, we obtain

$$\xi_t(\epsilon) \sim \mu/\epsilon. \tag{32}$$

At criticality and in the weakly disordered Griffiths phase, μ only provides logarithmic corrections to the leading $1/\epsilon$ dependence.

The same result can also be obtained more formally from the single-layer Langevin equation

$$\frac{\partial \phi(\mathbf{q},t)}{\partial t} = -2\Gamma_0(\epsilon + \gamma^2 \mathbf{q}^2)\phi(\mathbf{q},t) + \Gamma_0 \mu h(\mathbf{q},t) + \eta(\mathbf{q},t), \tag{33}$$

where $h(\mathbf{q},t)$ is a time-dependent magnetic field, $\eta(\mathbf{q},t)$ is the usual δ-correlated noise and Γ_0 fixes the overall time scale. To find the autocorrelation function of a single layer, we solve Eq. (33) for $h(\mathbf{q},t)=0$ and insert the solution into Eq. (31). In the asymptotic long-time limit, $\Gamma_0 \epsilon t \gg 1$, we find

$$C_l(t) \sim \exp(-2\Gamma_0 \epsilon t)/(\Gamma_0 \epsilon t), \tag{34}$$

in agreement with the heuristic estimate [Eq. (32)]. Solving the Langevin equation in the presence of a field allows us to calculate the single-layer dynamic susceptibility $\chi_l(\mathbf{q},\omega) = \partial m(\mathbf{q},\omega)/\partial h(\mathbf{q},\omega)$. For a uniform field, $\mathbf{q}=0$, this results in

$$\chi_l(\omega) = \mu^2/(2\epsilon - i\omega/\Gamma_0). \tag{35}$$

After having discussed the single-layer dynamics, we now turn to the full system. To find the average autocorrelation function at time t, we run the strong-disorder renormalization group to the scale $\Omega_t = 1/t$. All layers eliminated during this procedure have correlation times $\xi_t \ll t$ and do not contribute to the autocorrelation function. Surviving layers have $\xi_t \gg t$, they thus contribute proportional to their moment μ per site, giving $C(t) \sim n_{\Omega_t} \mu_{\Omega_t}$. At criticality, this leads to an ultraslow logarithmic decay of the autocorrelation function,

$$C(t) \sim [\ln(t/t_0)]^{\phi - 1/\psi}, \tag{36}$$

with t_0 as a microscopic time scale. In the weakly disordered Griffiths phase, the same calculation yields, up to logarithmic corrections, a nonuniversal power law

$$C(t) \sim \delta^{\nu + \nu\psi(1-\phi)} t^{-1/z}. \tag{37}$$

The same time dependence also follows from averaging Eq. (34) over the spectral density $\rho(\epsilon)$. The power-law decay [Eq. (37)] is much slower than the stretched exponential found in conventional classical Griffiths phases.[20–23] Interestingly, Eqs. (36) and (37) are reminiscent of the behavior at certain classical *nonequilibrium* phase transitions with disorder.[27–29]

The uniform dynamic susceptibility can be computed in an analogous manner. At criticality, we find its imaginary part to behave as

$$\chi''(\omega) \sim \frac{1}{\omega}[\ln(\Gamma_0/\omega)]^{2\phi - 1/\psi}. \tag{38}$$

In the weakly disordered Griffiths phase, we again obtain a power law,

$$\chi''(\omega) \sim \delta^{\nu + 2\nu\psi(1-\phi)} \omega^{1/z - 1}. \tag{39}$$

For the local dynamic susceptibility $\chi_{loc}(\omega)$, the corresponding relations are $(1/\omega)[\ln(\Gamma_0/\omega)]^{\phi - 1/\psi}$ and $\delta^{\nu + \nu\psi(1-\phi)}\omega^{1/z-1}$ at criticality and in the Griffiths phase, respectively.

At first glance, the above results for $C(t)$ and $\chi''_{loc}(\omega)$ appear to violate the fluctuation dissipation theorem which requires $\chi''_{loc}(\omega) = (\omega/2T)C(\omega)$, where $C(\omega)$ is the Fourier transform of the autocorrelation function. The reason for the disagreement is that the relaxation time [Eq. (32)] diverges for the largest rare regions (which correspond to effective layers with the smallest ϵ). Thus, the layers that dominate the long-time tail of $C(t)$ are not in equilibrium, and the fluctuation-dissipation theorem is not applicable. Technically, the disagreement is caused by the fact that Eq. (34) cannot be used for the layers with the smallest ϵ at any finite time.

VII. DISCUSSION AND CONCLUSIONS

In summary, we have investigated the phase transition in a three-dimensional randomly layered classical Heisenberg magnet. We have employed a strong-disorder renormalization group to show that the critical point is of unconventional infinite-randomness character. Somewhat surprisingly, the critical behavior can be found exactly, making our system one of the very few examples of three-dimensional systems with exactly known critical exponents. The critical point is accompanied by strong power-law Griffiths singularities (which are often called *quantum* Griffiths singularities because they generically occur in quantum systems but not in classical systems). In addition to the thermodynamics, we have also studied the critical dynamics within model A of the Hohenberg-Halperin classification. It is characterized by an ultraslow relaxation of the magnetic correlations at criticality as well as in the Griffiths phase.

Our findings can be related to a broader classification[9,14] of phase transitions with quenched disorder. This classification is based on the effective dimensionality of the defects or, equivalently, the rare regions. Three classes can be distinguished: (i) if the defect dimensionality is below the lower critical dimension d_c^- of the problem, the resulting critical point is conventional, and the Griffiths singularities are exponentially weak. (ii) If the defect dimensionality is exactly equal to the lower critical dimension, the critical point is of infinite-randomness type and accompanied by power-law "quantum" Griffiths singularities. (iii) Finally, if the defects are above the lower critical dimension, individual regions

can order independently, leading to a smeared transition.

The randomly layered Heisenberg magnet falls into class (ii) because the dimensionality of the planar defects is two, identical to the lower critical dimension of the classical Heisenberg model. The results of this paper are therefore in complete agreement with the general classification. It is worth noting, that the behavior of a randomly layered *Ising* magnet is very different. The lower critical dimension of a classical Ising model is one, thus planar defects are above the lower critical dimension. Consequently, the phase transition in a randomly layered Ising magnet is smeared by the disorder.[30,31]

We emphasize that the above classification also helps resolve the puzzling question posed at the end of Sec. III B, viz., why systems as different as the randomly layered Heisenberg magnet and the McCoy-Wu model (or, equivalently, the random transverse-field Ising chain) end up in the same universality class. The crucial point is that even though these two systems have different order parameter symmetries and dimensionalities, the defects are exactly at the lower critical dimension in both cases: $d_c^-=2$ for the classical Heisenberg model and $d_c^-=1$ for the Ising model. These arguments demonstrate why both the McCoy-Wu model and our randomly layered Heisenberg model end up having infinite-randomness critical points and thus the same scaling scenario; they do not yet explain why the two systems share the same critical exponent values. The agreement of the exponent *values* follows from the fact that both systems are random in one direction which leads to coarse graining in one dimension within the strong-disorder renormalization group. Moreover, the renormalization-group fixed point only depends on the multiplicative structure the recursion relations (9) and (10) and not on model-dependent prefactors.

Our explicit calculations have been performed in the large-N limit of the order parameter field theory [Eq. (6)]. However, the critical fixed point stays valid for all $N>2$ including the case of Heisenberg symmetry, $N=3$. To see this, we need to confirm that the recursion relations (9) and (10) remain unchanged for any $N>2$. The multiplicative structure of the recursion [Eq. (9)] for $J^\perp$ follows directly from second order perturbation theory and is thus the same for all N. In contrast, the multiplicative structure of the recursion relation (10) for the renormalized distance ϵ from criticality follows from the fact that a single layer of a continuous symmetry order parameter ($N>2$) is exactly at the lower critical dimension. This implies an exponential dependence of ϵ on the moment of the effective layer and thus the multiplicative form of Eq. (10); for details see the corresponding discussion in Ref. 19. Consequently, up to unimportant prefactors, both recursion relations remain valid for any $N>2$ and with them the infinite-randomness critical point scenario found in this paper.

The strong-disorder renormalization group allowed us to identify the infinite-randomness fixed point and verify its stability. However, it cannot tell whether or not a weakly or moderately disordered system will flow toward this fixed point. This is due to the fact that for weak disorder, the renormalization-group recursions are not very accurate. To gain further insight, it is useful to look at the behavior in the weak disorder limit. The effects of weak disorder on a clean critical point are governed by the Harris criterion[32] that states that the clean fixed point is stable against disorder if its correlation length exponent ν fulfills the inequality $d_r\nu>2$, where d_r is the number of dimensions in which there is randomness. In our case, $d_r=1$ and the correlation length exponent of the clean 3D Heisenberg model is[33] $\nu\approx 0.711$. Therefore, the clean 3D Heisenberg critical point violates the inequality $d_r\nu>2$ implying that it is unstable against weak planar disorder. [It is also unstable against linear disorder, $d_r=2$ (see Ref. 34), but stable against the usual point disorder, $d_r=3$.] Within a renormalization-group approach this means that weak planar disorder initially increases under renormalization suggesting that our fixed point may control the critical behavior for all bare disorder strength. A more complete answer to this question will likely come from experiment and computer simulations.

Experimental verifications of infinite-randomness critical behavior and the accompanying power-law "quantum" Griffiths singularities have been hard to come by, in particular in higher-dimensional systems. Only very recently, promising measurements have been reported[35,36] of the quantum phase transitions in $CePd_{1-x}Rh_x$ and $Ni_{1-x}V_x$. We hope that our work opens an alternative avenue to observe these phenomena in systems that may be easier to study experimentally.

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